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We present a theoretical study of the propagation of a plane pressure wave in an unbounded monodisperse bubbly liquid. At large wavelengths, this liquid is described as a continuous diphasic medium. Two passing bands are then displayed, separated by a forbidden band. The low-frequency branch, weakly affected by water compressibility, corresponds to a bubble wave propagation. The high-frequency branch, strongly dependent on water compressibility, accounts for sound propagation. Our results are compared with the dispersion relation obtained by Foldy in the mid forties, and are found in partial agreement with it. The role of dissipation is discussed.

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I. INTRODUCTION

The acoustic bubble is a tremendous sound scatterer. Hence its utmost importance in the acoustics of bubbly liquids, including everyday life manifestations like the sound of running water [1] or the “hot chocolate effect” [2]. From the early thirties on, an interesting challenge was to measure and calculate the sound velocity in bubble clouds. We briefly recall hereafter a few outstanding attempts in this aim, and take the opportunity to introduce some notations that we shall use in the next sections.

A first answer was proposed as soon as 1932 by Wood [3], with an effective medium model : f standing for the air volume fraction in the cloud, and indices a and w respectively referring to air and water, the effective mass density $\rho_{\rm eff} = f\rho_a + (1-f)\rho_w$ and compressibility $\chi_{\rm eff} = f\chi_a + (1-f)\chi_w$ are used to define the effective velocity

$$c_{\rm eff} = \frac{1}{\sqrt{\rho_{\rm eff}\chi_{\rm eff}}} = \frac{1}{\sqrt{(f\rho_a + (1-f)\rho_w)(f\chi_a + (1-f)\chi_w)}}, \tag{1.1a}$$

known as the Wood velocity. For small air volume fractions ($f \ll 1$), the effective medium has the mass density of the water ($\rho_{\rm eff} \simeq \rho_w$) and the compressibility $\chi_{\rm eff} = f\chi_a + \chi_w$, so that (1.1a) simplifies in

$$c_{\rm eff} \simeq \frac{1}{\sqrt{\rho_w(f\chi_a + \chi_w)}}. \tag{1.1b}$$

Wood’s model accounts for a most spectacular observation: the inclusion of minute quantities of gas in a liquid strongly modifies the acoustics of the latter. For example, a $f = 10^{-4}$ air volume fraction added to pure water lowers the sound velocity from 1500 m.s^{-1} to 900 m.s^{-1}: a 0.01 % change in f thus yields a 40 % change in c. In fact, Wood’s model predicts a nondispersive sound propagation and is available only for frequencies small compared to the individual bubbles’ Minnaert frequency [1]. About or beyond this frequency, the proper bubbles’ dynamics has to be taken into account.

In 1945, Foldy published an article on wave multiscattering [4] which still remains a reference today, and which has been applied [5] to the acoustic propagation through bubbly liquids. In the case of a monochromatic plane pressure wave $p(\vec r, t) = \Re\{P\,{\rm e}^{{\rm i}(\vec k.\vec r - \omega t)}\}$ propagating through a monodisperse bubble cloud, using Foldy’s theory yields

$$k^2 = k_{\rm s}^2 + 4\pi n\mathcal{S}. \tag{1.2}$$

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This formula is referred to as the (monodisperse) "Foldy's dispersion relation" throughout the present paper. In this formula, $k = k(\omega)$ is the effective wave vector, $k_\mathrm{s} = \omega/c_w$ is the sound wave vector in pure water, n the number of scatterers (*i.e.* bubbles under the circumstances) per unit volume and $\mathcal{S}$ the scattering function. It is important to underline that, in Foldy's theory, $\mathcal{S}$ is calculated as if the scattering bubble was *alone* in the cloud. As a consequence,

$$\mathcal{S} = \mathcal{S}(\omega, R) = \frac{R\omega^2}{\omega_0^2 - \omega^2 - \mathrm{i}\omega\mathit{\Gamma}}, \tag{1.3}$$

where R is the bubble's radius,

$$\omega_0 = \sqrt{\frac{3}{\chi_a \rho_w R^2}} \tag{1.4}$$

is its Minnaert angular frequency, and $\mathit{\Gamma}$ stands for its radiative damping rate. The calculation of $\mathit{\Gamma}$ and of further additional damping rates (viscous and thermal), as well as the choice of the gas compressibility (between isentropic and isothermal), has been discussed at some length by Prosperetti and can be found in [6]. By care of simplicity, and for reasons discussed at the end of section III, we (provisionally) deliberately disregard any damping.

The air volume fraction reads $f = \frac{4}{3}\pi n R^3 = \frac{4}{3}\pi\left(\frac{R}{d}\right)^3$, d being the average nearest-neighbour distance between bubbles. Let us define the critical wave vector

$$k_\mathrm{c} = \sqrt{\frac{3f}{R^2}}. \tag{1.5}$$

Observe that $k_\mathrm{c} \simeq \left(\frac{f}{2}\right)^{1/6} \times \frac{\pi}{d} < \frac{\pi}{d}$. With the above definition, Foldy's dispersion relation reads

$$\left(\omega^2 - \omega_+^2(k)\right)\left(\omega^2 - \omega_-^2(k)\right) = 0, \tag{1.6a}$$

with

$$\omega_\pm^2(k) = \frac{1}{2}\Big\{(k^2 + k_\mathrm{c}^2)c_w^2 + \omega_0^2 \pm \sqrt{[(k^2 + k_\mathrm{c}^2)c_w^2 + \omega_0^2]^2 - 4\omega_0^2 c_w^2 k^2}\Big\}. \tag{1.6b}$$

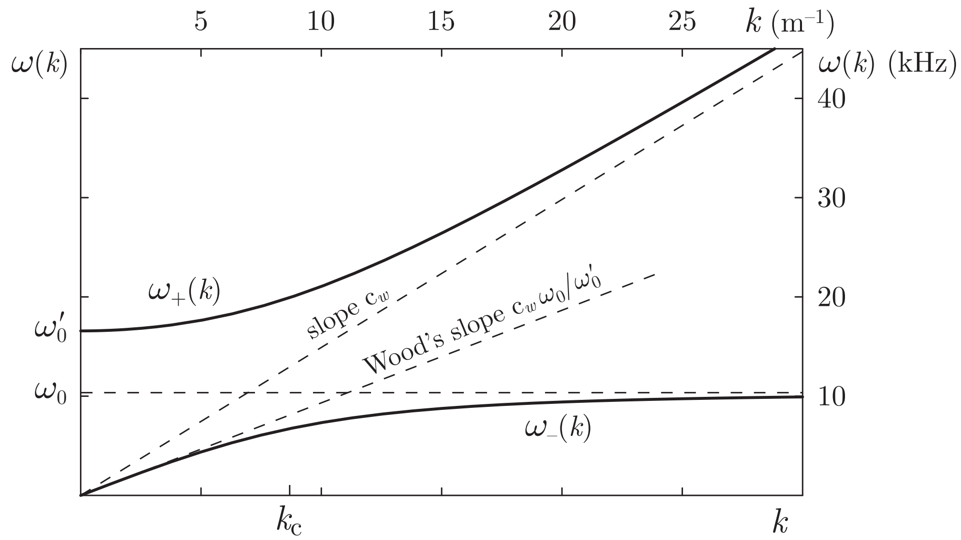

FIG. 1: Foldy's damping-free dispersion relation for a monodisperse bubbly liquid. Two branches are predicted, with the gap $[\omega_0, \omega_0' = \sqrt{\omega_0^2 + k_\mathrm{c}^2 c_w^2}]$. The lower branch starts linearly for $k \ll k_\mathrm{c}$ with a slope $c_w\omega_0/\omega_0'$ that coincides with the Wood velocity for dilute bubbles ($f \ll 1$), and it tends asymptotically towards the Minnaert angular frequency ω_0 of the bubbles. The upper branch starts at $\omega_+(0) = \omega_0'$ and ends up asymptotically with a Klein-Gordon behaviour: $\omega_+(k) \simeq \sqrt{(k^2 + k_\mathrm{c}^2)c_w^2}$. The numerical values correspond a bubble radius $R = 2$ mm and an air volume fraction $f = 0.01\,\%$ in water under atmospheric pressure. Hence a mean nearest-neighbour distance between bubbles $d = 70$ mm and a critical wave vector $k_\mathrm{c} = 8.7$ m^{-1}; note that $k_\mathrm{c} < \frac{\pi}{d}$. No wave can propagate in this bubbly water at angular frequencies comprised between $\omega_0 = 10$ kHz and $\omega_0' = 16$ kHz.

As displayed by the above formulas (1.6) and by figure 1, two branches are found, separated by a forbidden band. The lower branch $\omega_-(k)$ is nondispersive at large wavelengths ($k \ll k_\mathrm{c}$), with an effective velocity $c_w\omega_0/\sqrt{\omega_0^2 + k_\mathrm{c}^2 c_w^2} = 1/\sqrt{\rho_w(f\chi_a + \chi_w)}$, which coincides with Wood's velocity (1.1b) for $f \ll 1$. This lower branch has an horizontal asymptot : $\omega_-(k) \to \omega_0$ for $k \gg k_\mathrm{c}$. The upper branch starts at $\omega_+(0) = \sqrt{\omega_0^2 + k_\mathrm{c}^2 c_w^2} = \omega_0'$ (low cutoff), and shows an asymptotic behaviour of the Klein-Gordon type : $\omega_+(k) \simeq \sqrt{(k^2 + k_\mathrm{c}^2)c_w^2}$. No wave with angular frequency comprised between ω_0 and ω_0' can propagate in the bubbly water.

The above result (1.2) was obtained by Foldy in the framework of the mean field approximation (the average being performed over all the possible realizations of the bubble cloud), yet with a further simplification: the (average) pressure *undergone* by a given scatterer, say bubble j under the circumstances, and due to its neighbours, is assumed to be the *same* as if bubble j was *not* present in the cloud. In other words, the (average) action exerted on bubble j by its surroundings is calculated *as if* this surroundings was not "polarized" by bubble j. In terms of scattering paths, Foldy's simplification consists in neglecting all loop-diagrams (*i.e.* diagrams with one or several bubbles being visited twice or more by the scattering path). Now, as displayed by (1.3), two like bubbles (same radius) have their resonance for the same Minnaert angular frequency ω_0; thus, if close to each other, and for $\omega \simeq \omega_0$, they will bounce each other's emitted waves many times, thus weighting loop-diagrams' contribution. This situation is unavoidably found in a monodisperse cloud; the denser the bubble cloud, the bolder Foldy's approximation.

Accordingly, substantial discrepancies between experimental data and Foldy's predictions are likely to arise in the $\omega \simeq \omega_0$ zone of the dispersion relation, chiefly for dense monodisperse bubble clouds. As a matter of fact, such discrepancies were revealed as soon as 1957 with Silberman's experimental data [7]. A detailled list of other experiments can be found among the references of the article by Commander and Prosperetti [8]. More recent works performed in Roy's group [9–11] seem nevertheless to be in satisfactory accordance with Foldy's model for air volume fractions f up to $5\,10^{-4}$, but the authors' conclusion is discreet about it.

On the other hand, there is a wealth of theoretical contributions concerning sound propagation in bubbly liquids. Most of them found upon Foldy's above-cited pioneering work and apply themselves to complete it, as for instance the paper by Lax [12]. Waterman and Truell [13] have elaborated a validity criterion of Foldy's approximation (*i.e.* neglecting loop-diagrams). Commander and Prosperetti, in their 1989 paper [8], use a theory proposed by Caflisch *et al.* [14], and compare it with the experimental results. They conclude that "the model works very well up to (air) volume fractions of 1 % - 2 % provided that bubble resonances play a negligible role. Such is the case in a mixture of many bubble sizes or, when only one or a few sizes are present, away from the resonance frequency regions for these sizes. In the presence of resonance effects, the accuracy of the model is severly impaired." Besides, several authors have tackled the loop-scattering process problem. Feuillade [15, 16] proposed corrections to Foldy's formula, but his model is not beyond controversy [17–21]. By means of diagramatic methods used in particles or statistical physics, Ye and Ding [22] calculate a correction to Foldy's relation taking into account all one-loop diagrams, which ends up in adding a further complex part to the scattering function $\mathcal{S}$ displayed in (1.3), but turns out to be insufficient. Henyey [19] resumes the diagramatic study and implements a further step: his result consists *in fine* in calculating the radiative damping of the bubbles in an effective medium with wave vector k, instead of pure water with wave vector k_s. Kargl [23] obtains the same result with a different method : he resumes Commander and Prosperetti's calculation, now considering that the bubbles oscillate (not only radiate) in the afore-cited effective medium. But the discrepancy with Silberman's measurements holds in the $\omega \simeq \omega_0$ region ...

The above brief review of the state of the art is of course neither exhaustive nor accurate. But it shows that finding out a satisfactory dispersion relation for the propagation in bubbly liquids is still an open challenge. Any would-be candidate falling into step with Foldy's analysis should imperatively take all scattering loop processes into account, in order to remain available when bubbles' resonances play a major role, *i.e.* at $\omega \simeq \omega_0$ in dense monodisperse bubble clouds.

It is our ambition in the present paper to bring a piece of answer to this question. Disregarding Foldy's approach in terms of multiscattering, we choose an utterly different one: the bubble cloud is no longer considered as a set of individual scatterers, but as a whole nonlocalized system in which bubble-bubble interactions are taken into account from the very start of the calculation. For the sake of legibility, we *derive* the basic motion equations of the problem in the framework of the polydisperse cloud, but we focus on the monodisperse cloud case when *solving* them. In order to introduce and develop this approach, the present paper is organized as follows.

In section II, we consider a bubble cloud in incompressible water. The motion of water is calculated as a simple consequence of the bubbles' breathing and involves no further degree of freedom: a N-bubble cloud is thus a N-degree-of-freedom system, for which the calculation of water's kinetic energy allows to define a $N \times N$ mass matrix. Thereby, the bubbles have lost their individuality and merge in a whole, the large-wavelength eigenmodes of which are easily determined in the unbounded cloud limit case, and referred to as "bubble waves".

In section III, we take the water compressibility into account. To begin with (subsection III A), we outline what we call the "radiative picture": the water displacement $\vec{u}$ and extrapressure p fields are split into two parts. The former parts $(\vec{u}_{\rm i}, p_{\rm i})$ correspond to section II's incompressible water approximation, and are expressed as functions of the mere bubble dynamical variables. The latter parts $(\vec{u}_{\rm r}, p_{\rm r})$ account for the proper water degrees of freedom, and are associated with usual sound waves. We derive two coupled motion equations: that of the bubbles' breathings driven by the sound field, and reciprocally that of the sound field driven by the bubbles' breathings. Then (subsection III B), we look for monochromatic propagative solutions of the above set of coupled motion equations, and derive a dispersion relation which, amazingly, coincides with Foldy's (damping-free) one. This unexpected coincidence is discussed.

From section IV on, we definitely focus on the large wavelength (in a sense precised in the text) propagation modes. With this aim, we substitute macroscopic (averaged) radiative fields $(\vec{U}_{\rm r}, P_{\rm r})$ for the microscopic ones $(\vec{u}_{\rm r}, p_{\rm r})$, and we introduce a continuous field to describe the (discrete) bubble's breathings. The coupled motion equations derived in subsection III A are reconsidered in the Fourier $\vec{k}$-space, and the dispersion relation is recovered whereas the description of the propagation modes corresponding to both branches of the dispersion curve is improved by degrees.

In section V we address two – thitherto disregarded – questions: (i) how a usual pressure source (loud-speaker or else) does excite the propagative modes (subsection V A)? We show that bubble waves and sound waves are simultaneously excited and can be determined by means of a unique Green function. (ii) How is dissipation accounted for in the radiative picture (subsection V B)? We finally take thermal and viscous processes into account and complete the dispersion relation accordingly.

Miscellaneous comments and remarks are gathered in the conclusion section VI.

II. BUBBLE WAVES

Let us consider a set of spherical gas (say air) bubbles in a unbounded liquid (say water). At breathing equilibrium under pressure P_0 and temperature T_0, bubble j is supposed to be trapped at point $\vec{r}_j$ and its radius is R_j. In the course of the motion, this radius becomes $R_j(1+x_j(t))$. Let $\vec{r}+\vec{u}(\vec{r},t)$ be the position at time t of the water element that rests at point $\vec{r}$ at equilibrium: $\vec{u}(\vec{r},t)$ is therefore the displacement field in the so-called Lagrange's picture. If water was incompressible – which we shall assume throughout this section II – $\vec{u}(\vec{r},t)$ would read, at the linear approximation,

$$\vec{u}(\vec{r},t) = \sum_j R_j^3 \vec{\mu}_j(\vec{r}) x_j(t), \quad \text{with } \vec{\mu}_j(\vec{r}) = \frac{\vec{r}-\vec{r}_j}{|\vec{r}-\vec{r}_j|^3}. \tag{2.1}$$

With this displacement field is associated, neglecting air's inertia, the total kinetic energy

$$E_{\rm kin} = \int {\rm d}^3 r \ \frac{1}{2}\rho_w \left(\frac{\partial \vec{u}}{\partial t}\right)^2 = \frac{1}{2}\sum_j \sum_{j'} R_j^3 R_{j'}^3 m_{jj'} \dot{x}_j \dot{x}_{j'}, \tag{2.2a}$$

with

$$m_{jj'} = \int {\rm d}^3 r \ \rho_w \vec{\mu}_j . \vec{\mu}_{j'} . \tag{2.2b}$$

The above integrals range over the volume occupied by the water. Nevertheless, if the condition $f \ll 1$ is fulfilled (which will be assumed throughout this paper), the integration volume can be extended as follows.

(i) For the calculation of the diagonal term m_{jj}, one integrates over all space outside bubble j (*i.e.* as if bubble j was alone), which yields

$$m_{jj} = 4\pi \frac{\rho_w}{R_j}. \tag{2.3a}$$

Multiplying the above result by R_j^4, we recover the well known Minnaert mass $M_j = 4\pi\rho_w R_j^3$.

(ii) For the calculation of the off-diagonal terms, a remarkable result is that $m_{jj'}$ $(j \neq j')$ is independent of radii R_j and $R_{j'}$. Ignoring the other bubbles (and thus integrating over the whole space), one finds

$$m_{jj'} = 4\pi \frac{\rho_w}{|\vec{r}_j - \vec{r}_{j'}|}. \tag{2.3b}$$

On the other hand, since the water is assumed to be incompressible, all the elastic potential energy of the system originates in the gas compression (the contribution of the surface tension energy is disregarded here by care of simplicity, but it can be exactly taken into account, as explained below), and consequently reads

$$E_{\mathrm{pot}} = \frac{1}{2}\sum_j R_j^3 \frac{12\pi}{\chi_a} x_j^2. \tag{2.4}$$

Thus, neglecting any kind of dissipation, we can build the Lagrangian $L_{\mathrm{b}}(\{x_j, \dot{x}_j\}) = E_{\mathrm{kin}} - E_{\mathrm{pot}}$ and derive the set of motion equations

$$\sum_{j'} R_{j'}^3 m_{jj'} \ddot{x}_{j'} + \frac{12\pi}{\chi_a} x_j = 0 \quad (\forall j). \tag{2.5}$$

Finding the eigenmodes and eigenfrequencies of the above system is theoretically possible with a computer: for a N-bubble cloud, one has to diagonalize a $N \times N$ symmetrical real matrix. For an infinite homogeneous cloud, one can look for monochromatic solutions of the form

$$x_j(t) = \Re\{\overline{X}\, \mathrm{e}^{\mathrm{i}(\vec{k}.\vec{r}_j - \omega t)}\}, \tag{2.6}$$

where $\overline{X}$ is a complex amplitude. In the case of simple (*i.e.* with one bubble per unit cell) bubble-crystals, the exact solution (2.6) can actually be found whatever the wave vector $\vec{k}$ (chosen in the first Brillouin zone, as usual). With such bubble-crystals, separating the $j = j'$ and $j \neq j'$ terms, factorizing by $4\pi\rho_w R^2$ and introducing the Minnaert angular frequency ω_0 displayed in (1.4), equation (2.5) reads

$$\ddot{x}_j + \sum_{j' \neq j} \frac{R}{|\vec{r}_j - \vec{r}_{j'}|} \ddot{x}_{j'} + \omega_0^2 x_j = 0, \tag{2.7a}$$

and admits solution (2.6) provided that

$$\omega^2 = \omega_{\mathrm{b}}^2(\vec{k}) = \frac{\omega_0^2}{1 + \displaystyle\sum_{\vec{s} \neq 0} \frac{R}{|\vec{s}|} \mathrm{e}^{\mathrm{i}\vec{k}.\vec{s}}}, \tag{2.7b}$$

index b in ω_{b} standing for "bubble wave" and $\sum_{\vec{s} \neq 0}$ for the summation over all bubble sites except the origin.

In the case of bubble-glasses (*i.e.* disordered clouds), expression (2.6) is not an exact solution of (2.5). Nevertheless, at wavelengths large compared to the nearest neighbours mean bubble distance d (in other words for wave vectors $\vec{k}$ situated in the center of the Brillouin zone: $|\vec{k}| \ll \frac{\pi}{d}$), bubble waves make no difference between ordered or disordered clouds. Then, substituting the integral $\frac{3f}{4\pi}\int \mathrm{d}^3 r_j$ for the discrete sum $\sum_{j'} R_{j'}^3$, the dispersion relation (2.7b) becomes

$$\omega_{\mathrm{b}}^2(\vec{k}) = \frac{\omega_0^2}{1 + \dfrac{k_{\mathrm{c}}^2}{k^2}}, \quad k \ll \frac{\pi}{d}, \tag{2.7c}$$

with the critical wave vector k_{c} defined in (1.5). The above bubble wave dispersion relation is displayed in dotted line in figure 2. Two different regimes can be distinguished.

(*i*) In the $k \ll k_{\mathrm{c}}$ region, formula (2.7c) can be approximated by $\omega_{\mathrm{b}}(\vec{k}) \simeq \omega_0 k/k_{\mathrm{c}}$: bubble wave propagation is nondispersive. The associated velocity $\omega_0/k_{\mathrm{c}} = 1/\sqrt{f\chi_a\rho_w}$ – which depends only, for given χ_a and ρ_w, on the air volume fraction f (*not* on the bubbles' sizes) – is in accordance with expression (1.1b) where χ_w is set equal to zero (incompressible water). In this region, the dynamics of the cloud is dominated by bubble-bubble interactions: in the left-hand side of equation (2.7a), the $\ddot{x}_j$ term is negligible compared to the $\sum_{j' \neq j} \ddot{x}_{j'}$ term.

(*ii*) In the $k \gg k_{\mathrm{c}}$ region, formula (2.7c) can be approximated by $\omega_{\mathrm{b}}(\vec{k}) \simeq \omega_0$: bubble wave propagation is dispersive. In this region, the dynamics of the cloud is dominated by individual Minnaert bubble oscillations: the $\sum_{j' \neq j} \ddot{x}_{j'}$ bubble-bubble interaction term is negligible compared to the individual $\ddot{x}_j$ term in (2.7a).

The $k \simeq k_\mathrm{c}$ region corresponds to the crossover between the two above regimes, so that k_c^{-1} may be regarded as an *effective* bubble-bubble interaction range.

Besides, as announced above, it is easy to take capillarity effects into account. A well known calculation [24] shows indeed that expression (2.4) of the potential energy should be completed in

$$E_\mathrm{pot} = \frac{1}{2}\sum_j R_j^3 \frac{12\pi}{\chi_a}\Big(1+\frac{2\sigma}{RP_0}\frac{3\gamma-1}{3\gamma}\Big)x_j^2, \tag{2.8a}$$

where σ and γ respectively stand for the air/water surface tension and the air heat capacities ratio. As a consequence, equations (2.7) are still valid, provided that expression (1.4) of the Minnaert angular frequency ω_0 is changed in

$$\omega_0 = \sqrt{\frac{3}{\chi_a \rho_w R^2}\Big(1+\frac{2\sigma}{RP_0}\frac{3\gamma-1}{3\gamma}\Big)}. \tag{2.8b}$$

With $\sigma = 7\,10^{-2}\,\mathrm{J.m}^{-2}$ and under atmospheric pressure $P_0 = 10^5$ Pa, the ratio $\frac{2\sigma}{RP_0}$ equals unity for $R = 1.4$ µm: the above corrections (2.8) can be omitted for bubbles radii R larger than 0.1 mm. For the sake of simplicity, we shall henceforth ignore capillarity effects; of course, they can be ultimately taken into account thanks to formulas (2.8) if small bubbles ($R \lesssim 0.1$ mm) are considered.

The discussion of the eigenmodes of bubble clouds in incompressible water is far from finished. For instance, one could study the dispersion relation of bubble waves in complex bubble-crystals: with two (or more) bubbles par unit cell (with possibly different radii), two (or more) branches should be found, as in polyatomic lattice dynamics. In another connection, localized modes are likely to be found about isolated "defects" (*i.e.* bubbles with anomalous radii or positions) in bubble-crystals, or in bubble-glasses. Interesting though it may be, this discussion is outside the scope of the present paper devoted to mere wave propagation.

In the next section, we shall focus on water's proper degrees of freedom.

III. SOUND WAVES

Water is *not* incompressible. Although far much harder than air ($\chi_w \simeq 4.5 \times 10^{-10}$ Pa^{-1} $\ll \chi_a \simeq 7 \times 10^{-6}$ Pa^{-1} at atmospheric pressure), it has its own degrees of freedom. Due to the associated finite velocity c_w of pressure waves, retardation effects corrections should be taken into account in the bubble-bubble interactions that where considered as instantaneous in section II. This is the aim of the present section III. To begin with, let us briefly outline what we have coined [25] the "radiative picture".

A. The radiative picture

Let us consider one bubble with radius $R\,(1+x(t))$ at the origin of coordinates in an unbounded extent of water. Let $P(\vec r, t) = P_0 + p(\vec r, t)$ be the total pressure at time t of the water element that rests at point $\vec r$ at equilibrium ($p(\vec r, t)$ is therefore the extrapressure field in the Lagrange's picture). It can be shown [25] that the extrapressure and the displacement generated at point $\vec r$ in the water respectively read

$$p(\vec r, t) = \rho_w \frac{R^3}{r}\Big(1+\frac{R}{c_w}\frac{\partial}{\partial t}\Big)^{-1}\ddot x\big(t-\frac{r-R}{c_w}\big), \tag{3.1a}$$

$$\vec u(\vec r, t) = \frac{R^3}{r^2}\Big(1+\frac{r}{c_w}\frac{\partial}{\partial t}\Big)\Big(1+\frac{R}{c_w}\frac{\partial}{\partial t}\Big)^{-1} x\big(t-\frac{r-R}{c_w}\big)\vec e_r \tag{3.1b}$$

(where $\vec e_r = \vec r/r$ is the unit vector from the bubble's center). If the water was incompressible, as assumed in section II, the above expressions would simplify in

$$p_\mathrm{i}(\vec r, t) = \rho_w \frac{R^3}{r}\ddot x(t), \tag{3.2a}$$

$$\vec u_\mathrm{i}(\vec r, t) = \frac{R^3}{r^2}x(t)\vec e_r = R^3 x(t)\,\vec\mu(\vec r), \tag{3.2b}$$

where index i indifferently stands for "incompressible" or "instantaneous". In the small air volume fractions ($f \ll 1$) limit we consider in this paper, and at the linear approximation, expressions (3.1a) through (3.2b) can simply be summed up over all the bubbles to obtain the extrapressure and displacement fields generated by a bubble cloud (so did we to derive (2.1) for instance). Note by the way that, according to equations (3.2), one has

$$\rho_w \frac{\partial^2 \vec{u}_\mathrm{i}}{\partial t^2} = -\vec{\nabla} p_\mathrm{i}, \tag{3.3a}$$

which, not surprisingly, means that Newton's equation is still valid at the $c_w \to \infty$ limit. The above result may also be interpreted as follows : in the water element volume $\mathrm{d}^3 r$'s accelerated frame, there prevails an apparent gravitational field $\vec{g}_\mathrm{app} = -\frac{\partial^2 \vec{u}_\mathrm{i}}{\partial t^2}$ that involves a hydrostatic pressure gradient equal to $\rho_w \vec{g}_\mathrm{app}$. Since water at infinity is still and at the equilibrium pressure P_0, $p_\mathrm{i}(\vec{r}, t)$ is easily calculated using (2.1) and integrating (3.3a), which yields

$$p_\mathrm{i}(\vec{r}, t) = \rho_w \sum_j R_j^3 \frac{\ddot{x}_j(t)}{|\vec{r} - \vec{r}_j|}, \tag{3.3b}$$

so that result (3.2a) is recovered, and extended to a multi-bubble source. Observe too that $\Delta\, p_\mathrm{i} = 0$ in the water. Moreover, it will reveal in the following to be convenient to split p and $\vec{u}$ into their instantaneous (i) and retarded (r) parts:

$$p(\vec{r}, t) = p_\mathrm{i}(\vec{r}, t) + p_\mathrm{r}(\vec{r}, t), \tag{3.4a}$$

$$\vec{u}(\vec{r}, t) = \vec{u}_\mathrm{i}(\vec{r}, t) + \vec{u}_\mathrm{r}(\vec{r}, t). \tag{3.4b}$$

The above splitting (3.4) deserves a few comments. First, as a consequence of general Newton's law $\rho_w \frac{\partial^2 \vec{u}}{\partial t^2} = -\vec{\nabla} p$ and of its instantaneous declension (3.3a), one has by subtraction

$$\rho_w \frac{\partial^2 \vec{u}_\mathrm{r}}{\partial t^2} = -\vec{\nabla} p_\mathrm{r}. \tag{3.5}$$

We would emphasize that the above-defined radiative extrapressure p_r is *not* equal to $-\frac{1}{\chi_w} \operatorname{div} \vec{u}_\mathrm{r}$. Since $\operatorname{div} \vec{u}_\mathrm{i} = 0$ in the water, the latter quantity coincides with $-\frac{1}{\chi_w} \operatorname{div} \vec{u}$, *i.e.* the *total* extrapressure p_w in the water, *including* the instantaneous extrapressure p_i generated by the bubbles as well as their acoustic radiation. Moreover, whereas the instantaneous fields p_i and $\vec{u}_\mathrm{i}$ are but linear combinations of the bubbles' dynamical variables x_j, as displayed by expressions (2.1) and (3.3b), the retarded fields p_r and $\vec{u}_\mathrm{r}$ (which, by definition, do cancel at the $\chi_w \to 0$ limit) involve water's degrees of freedom. It is noteworthy that the above splitting (3.4) is neither original nor specific to the acoustic problem under consideration. In the atomic physics domain, when regarding a set of electric charges – say an atom (or a molecule) – in interaction with an electromagnetic (EM) wave, a common and well-tried approach consists in choosing a Coulomb's gauge $\{\vec{A}, \varPhi\}$ to describe the EM field. Then, due to Coulomb's gauge condition $\operatorname{div} \vec{A} = 0$, the scalar potential obeys the electrostatic (*i.e.* instantaneous) Poisson's equation, and the electric field $\vec{E}$ is *ipso facto* split into an electrostatic (instantaneous) part $\vec{E}_\mathrm{i} = -\vec{\nabla}\varPhi$ and a radiative (retarded) part $\vec{E}_\mathrm{r} = -\frac{\partial \vec{A}}{\partial t}$. As well known, the *cohesion* of the atomic edifice – regarded as a whole as concerns the calculation of its energy levels – is mostly ensured by electrostatic forces (thus bringing part $\vec{E}_\mathrm{i}$ of the electric field into play), and its *interaction* with an outer EM wave, as well as the taking into account of retardation effects, is ruled by the radiative forces (involving part $\vec{E}_\mathrm{r}$ of the total electric field). In the vicinity of the atom, say a few angströms from the nucleus, the electrostatic field is, in most situations, orders of magnitude larger than the radiative field. Consequently, the latter should be considered as a small perturbation of the former, and treated accordingly. The approach we carry in the present paper strongly draws its inspiration from the above-recalled electrodynamics paradigm: our splitting (3.4) is but the acoustic adaptation of the EM splitting $\vec{E} = \vec{E}_\mathrm{i} + \vec{E}_\mathrm{r}$ in the Coulomb's gauge, also called "radiative gauge".

Let us now derive the equations of motion in the acoustic radiative picture. Neglecting air's inertia, subtracting E_pot (the bubbles' elastic potential energy (2.4)) from the full water Lagrangian $L_w = \frac{1}{2} \int \mathrm{d}^3 r \Big\{ \rho_w \big(\frac{\partial \vec{u}}{\partial t}\big)^2 - \frac{1}{\chi_w} (\operatorname{div} \vec{u})^2 \Big\}$, using (3.4b), (2.1) and $\operatorname{div} \vec{u}_\mathrm{i} = 0$, one gets

$$L = L_r + L_{rb} + L_b, \tag{3.6a}$$

where

$$L_\mathrm{r} = \frac{1}{2} \int \mathrm{d}^3 r \Big\{ \rho_w \Big(\frac{\partial \vec{u}_r}{\partial t} \Big)^2 - \frac{1}{\chi_w} (\operatorname{div} \vec{u}_r)^2 \Big\} \tag{3.6b}$$

is the radiative field's Lagrangian,

$$L_{\mathrm{b}} = \frac{1}{2}\sum_j \sum_{j'} R_j^3 R_{j'}^3 m_{jj'} \dot{x}_j \dot{x}_{j'} - \frac{1}{2}\sum_j R_j^3 \frac{12\pi}{\chi_a} x_j^2 \tag{3.6c}$$

is the bubble cloud's Lagrangian, and

$$L_{\mathrm{rb}} = \sum_j R_j^3 \dot{x}_j \int \mathrm{d}^3 r \; \rho_w \vec{\mu}_j . \frac{\partial \vec{u}_{\mathrm{r}}}{\partial t} \tag{3.6d}$$

is the {bubble cloud - radiative field} interaction Lagrangian. The integrals in the above expressions (3.6b) and (3.6d) range over the water extent.

From formulas (3.6), a rather straightforward calculation gives the motion equations.

(i) For the bubble cloud driven by the radiative field :

$$\sum_{j'} R_{j'}^3 m_{jj'} \ddot{x}_{j'} + \frac{12\pi}{\chi_a} x_j = -\int \mathrm{d}^3 r \; \rho_w \vec{\mu}_j . \frac{\partial^2 \vec{u}_{\mathrm{r}}}{\partial t^2}, \tag{3.7a}$$

which completes (2.5).

(ii) For the radiative field driven by the bubble cloud :

$$\frac{\partial^2 \vec{u}_{\mathrm{r}}}{\partial t^2} - c_w^2 \vec{\nabla}(\mathrm{div}\, \vec{u}_{\mathrm{r}}) = -\sum_j R_j^3 \ddot{x}_j \vec{\mu}_j . \tag{3.7b}$$

It is noteworthy that the above set of both equations (3.7), which couples dynamical variables $\{x_j\}$ and $\{\vec{u}_{\mathrm{r}}\}$, can be substituted by an equivalent set which couples $\{x_j\}$ and $\{p_{\mathrm{r}}\}$. To establish the latter set, we can simply consider the pressure splitting (3.4a) as explained hereafter.

(i) At bubble j's surface, the *total* extrapressure is $p(\vec{r}_j) = -\frac{3}{\chi_a} x_j$, whereas the *instantaneous* extrapressure is, owing to (3.3b), $p_{\mathrm{i}}(\vec{r}_j) = \rho_w \sum_{j'} R_{j'}^3 \frac{\ddot{x}_{j'}(t)}{|\vec{r}_j - \vec{r}_{j'}|}$. Applying (3.4a) at site $\vec{r}_j$, we get

$$\rho_w \sum_{j'} R_{j'}^3 \frac{\ddot{x}_{j'}}{|\vec{r}_j - \vec{r}_{j'}|} + \frac{3}{\chi_a} x_j = -p_{\mathrm{r}}(\vec{r}_j), \tag{3.8a}$$

the equivalence with (3.7a) of which is easily shown by means of an (Ostrogradski) integration by parts of the right-hand side, using (3.5).

(ii) Applying d'Alembert's equation $\frac{\partial^2 p}{\partial t^2} = c_w^2 \, \Delta \, p$ to splitting (3.4a), and allowing for (3.3b) and $\Delta \, p_{\mathrm{i}} = 0$, we get

$$\frac{\partial^2 p_{\mathrm{r}}}{\partial t^2} - c_w^2 \, \Delta \, p_{\mathrm{r}} = -\frac{\partial^2 p_{\mathrm{i}}}{\partial t^2} = -\rho_w \sum_j R_j^3 \frac{\ddddot{x}_j}{|\vec{r} - \vec{r}_j|}, \tag{3.8b}$$

which is equivalent to (3.7b), thanks to (3.5).

Let us summarize the situation as displayed by the above equations sets (3.7) or (3.8): we have two systems, namely the bubble cloud and the radiative field, driving each other's dynamics. In absence of any radiative field (*i.e.* $\vec{u}_{\mathrm{r}} = 0$, $p_{\mathrm{r}} = 0$), equations (3.7a) or (3.8a) simplify into (2.5): only the bubble cloud's modes are excited. If the latter modes are off (*i.e.* if $\{x_j\} = 0 \rightsquigarrow p_{\mathrm{i}} = 0$), equations (3.7b) or (3.8b) are then those of a simple d'Alembert's propagation, with velocity c_w and dispersion relation

$$\omega_{\mathrm{s}}(\vec{k}) = c_w |\vec{k}|, \tag{3.9}$$

index s in ω_{s} standing for "sound". For this reason, the radiative fields $(\vec{u}_{\mathrm{r}}, p_{\mathrm{r}})$ should be assimilated to what is commonly regarded as the *sound*. The sound dispersion relation $\omega_{\mathrm{s}}(\vec{k})$ is displayed in dashed lines in figures 1 and 2. In the next subsection, we shall seek at which condition bubble waves and sound can simultaneously propagate in a bubbly liquid.

B. The dispersion relation

Let us consider the set of coupled equations (3.8) and proceed as in section II: we start with a simple bubble-crystal and derive an exact dispersion relation, which we assume to be relevant for any homogenous bubble-glass if we limit to the first Brillouin zone's center ($|\vec{k}| \ll \pi/d$). For the bubble cloud, we thus look for monochromatic solutions of the form (2.6). For the radiative field, we look for monochromatic solutions of the form

$$p_{\rm r}(\vec{r},t) = \Re\big\{\overline{P}(\vec{r})\,{\rm e}^{{\rm i}(\vec{k}.\vec{r}-\omega t)}\big\}, \tag{3.10}$$

where the complex amplitude $\overline{P}(\vec{r})$ has the periodicity of the lattice. Thus equation (3.8a) reads, setting $\vec{r}_{j'} - \vec{r}_j = \vec{s}$:

$$\Big[\omega^2\Big(1+\sum_{\vec{s}\neq 0}\frac{R}{|\vec{s}|}{\rm e}^{{\rm i}\vec{k}.\vec{s}}\Big) - \omega_0^2\Big]\overline{X} = \frac{\overline{P}(\vec{r}_j)}{\rho_w R^2}. \tag{3.11}$$

On the other hand, as a consequence of its periodicity, $\overline{P}(\vec{r})$ reads as the Fourier series

$$\overline{P}(\vec{r}) = \sum_{\vec{K}} b_{\vec{K}}\,{\rm e}^{{\rm i}\vec{K}.\vec{r}}, \tag{3.12a}$$

where the summation ranges over all the wave vectors $\vec{K}$ of the reciprocal lattice and with

$$b_{\vec{K}} = \frac{1}{\mathcal{V}}\int {\rm d}^3r\,\overline{P}(\vec{r})\,{\rm e}^{-{\rm i}\vec{K}.\vec{r}}, \tag{3.12b}$$

the integral ranging over an arbitrary (integer) number of unit cells, with total volume $\mathcal{V}$. Then, using (3.10) and (2.6), the dynamical equation (3.8b) reads, all simplifications carried out,

$$\big[\omega^2 + c_w^2(-\,k^2 + 2{\rm i}\vec{k}.\vec{\nabla} + \Delta)\big]\overline{P}(\vec{r}) = \rho_w\omega^4\overline{X}f(\vec{r}) \tag{3.13a}$$

with

$$f(\vec{r}) = \sum_{j'} R^3\frac{{\rm e}^{{\rm i}\vec{k}.(\vec{r}_{j'}-\vec{r})}}{|\vec{r}_{j'}-\vec{r}|}. \tag{3.13b}$$

The above function $f(\vec{r})$ has the periodicity of the lattice and consequently reads

$$f(\vec{r}) = \sum_{\vec{K}} c_{\vec{K}}\,{\rm e}^{{\rm i}\vec{K}.\vec{r}} \tag{3.14a}$$

with

$$c_{\vec{K}} = \frac{1}{\mathcal{V}}\int {\rm d}^3r\,f(\vec{r})\,{\rm e}^{-{\rm i}\vec{K}.\vec{r}}. \tag{3.14b}$$

Then, using (3.12a) and (3.14a) in equation (3.13a), we get, for each wave vector $\vec{K}$:

$$\big[\omega^2 - \omega_{\rm s}^2(\vec{k}+\vec{K})\big]b_{\vec{K}} = \rho_w\omega^4\overline{X}c_{\vec{K}}. \tag{3.15}$$

Next, using the value of $b_{\vec{K}}$ derived from the above formula (3.15), we obtain $\overline{P}(\vec{r})$ thanks to (3.12a). In particular, we have the radiative pressure amplitude $\overline{P}(\vec{r}_j)$ undergone by bubble j:

$$\overline{P}(\vec{r}_j) = \sum_{\vec{K}}{\rm e}^{{\rm i}\vec{K}.\vec{r}_j}\frac{\rho_w\omega^4\overline{X}}{\omega^2-\omega_{\rm s}^2(\vec{k}+\vec{K})}\frac{1}{\mathcal{V}}\int {\rm d}^3r\,{\rm e}^{-{\rm i}\vec{K}.\vec{r}}\sum_{j'}R^3\frac{{\rm e}^{{\rm i}\vec{K}.(\vec{r}_{j'}-\vec{r})}}{|\vec{r}_{j'}-\vec{r}|}. \tag{3.16a}$$

This cumbersome formula fortunately simplifies, permuting $\sum_{j'}$ and $\int {\rm d}^3r$ (ranging over all space) and setting $\vec{s} = \vec{r}_{j'} - \vec{r}_j$, in

$$\overline{P}(\vec{r}_j) = \sum_{\vec{K}}\frac{\rho_w\omega^4\overline{X}}{\omega^2-\omega_{\rm s}^2(\vec{k}+\vec{K})}\times\frac{4\pi R^3}{(\vec{k}+\vec{K})^2}\frac{1}{\mathcal{V}}\sum_{\vec{s}}{\rm e}^{-{\rm i}\vec{K}.\vec{s}}, \tag{3.16b}$$

which can be furthermore simplified, noticing that $\vec{K}.\vec{s}$ is necessarily a multiple of 2π and that $\frac{1}{\mathcal{V}}\sum_{\vec{s}} = n$ (the number of bubbles per unit volume). Lastly, comparing (3.11) and the above result (3.16b), substituting k_c^2 for $4\pi nR$ (see (1.5)) and introducing the angular frequency $\omega_\mathrm{b}(\vec{k})$ defined in (2.7b), we get

$$\omega^2 - \omega_\mathrm{b}^2(\vec{k}) = \frac{\omega^4}{1 + \sum\limits_{\vec{s}\neq 0} \frac{R}{|\vec{s}|} \mathrm{e}^{\mathrm{i}\vec{k}.\vec{s}}} \sum_{\vec{K}} \frac{k_\mathrm{c}^2}{(\vec{k}+\vec{K})^2} \times \frac{1}{\omega^2 - \omega_\mathrm{s}^2(\vec{k}+\vec{K})}. \tag{3.17}$$

This is the exact dispersion relation in a (simple) bubble-crystal. It is noteworthy that all Bragg scattering processes are taken into account in the above formula. Nevertheless, in the center of the first Brillouin zone (*i.e.* for $|\vec{k}|$ much smaller than any (nonzero) $|\vec{K}|$) and for angular frequencies $\omega \ll c_w \frac{\pi}{d}$, the $\vec{K} = 0$ process prevails: retaining the mere $\vec{K} = 0$ term in the above summation over $\vec{K}$, formula (3.17) can be simplified in

$$\big(\omega^2 - \omega_\mathrm{b}^2(\vec{k})\big)\big(\omega^2 - \omega_\mathrm{s}^2(\vec{k})\big) = \omega^4 \frac{k_\mathrm{c}^2}{k_\mathrm{c}^2 + k^2}, \tag{3.18}$$

with ω_b given by approximation (2.7c). As explained above, (3.18) is also the dispersion relation in a monodisperse bubble-glass, provided of course that condition $|\vec{k}| \ll \pi/d$ is fulfilled. But relation (3.18) deserves a much surprising comment: it *exactly* coincides with (1.6a), the monodisperse-cloud damping-free Foldy's dispersion formula $\omega(\vec{k})$ displayed in figure 1. This coincidence is fascinating for the following reason: within the framework of the radiative picture introduced in subsection III A, the displacement (resp. pressure) radiated by the bubble is, *by construction*, incorporated in $\vec{u}_\mathrm{r}$ (resp. p_r). In other words, the bubbles' radiation is fully taken into account in our radiative picture (and does not involve any damping). We conclude that, in the denominator of Foldy's scattering function $\mathcal{S}$ recalled in (1.3), the imaginary damping term $-\mathrm{i}\omega\Gamma$ is superfluous. In fact, the acoustic radiation involves some damping only for isolated bubbles, more precisely in situations where the average distance d between two neighbouring bubbles is of the order or larger than the emitted acoustic wavelength. In the case we consider in this paper (*i.e.* in the center of the Brillouin zone), d is much smaller than the acoustic wavelength, and the *total* radiated power is by no means the sum of the powers that the bubbles would *individually* radiate. As a matter of fact, due to destructive interferences, there is no bubble radiation at all (except in directions $\vec{k}$ and $-\vec{k}$), and consequently no radiative damping. Damping occurs only when *true* dissipation (thermal or viscous) is taken into account, as explained in subsection V B.

In the next section, we shall apply ourselves to a discussion of the propagation modes.

IV. PROPAGATIVE MODES: CHARACTERIZATION

From now on, let us definitely restrict our study to the $\{|\vec{k}| \ll \frac{\pi}{d}, \omega \ll \frac{c_w\pi}{d}\}$ domain of the $\{\vec{k}, \omega\}$ plane. In the latter domain, as mentioned above, propagation makes no difference between ordered or disordered bubble clouds: the spatial variations of $\vec{u}_\mathrm{r}$ (resp. p_r) with wavelength of the order of (or smaller than) d are smoothed out, resulting in a "macroscopic" field $\vec{U}_\mathrm{r}$ (resp. P_r). Besides, the discrete set of bubble dynamical variables $\{x_j(t)\}$ can be substituted by a continuous scalar field $X(\vec{r}, t)$. The bubbly liquid can consequently be regarded as a continuous diphasic medium, the dynamics of which is described by means of a twofold macroscopic field $\{X(\vec{r}, t), \vec{U}_\mathrm{r}(\vec{r}, t)$ (resp. $P_\mathrm{r}(\vec{r}, t)\}$. Now it is noteworthy that the averaging that leads from the microscopic fields to the macroscopic ones is *ipso facto* achieved when calculating the spatial Fourier transform of the former fields at wave vectors $\vec{k}$ such that $|\vec{k}| \ll \frac{\pi}{d}$ (*i.e.* at the center of the Brillouin zone). To begin with, we shall turn this remark to profit and present a simple derivation of the motion equations ruling $X(\vec{k}, t)$ and $\vec{U}_\mathrm{r}(\vec{k}, t)$ (resp. $P_\mathrm{r}(\vec{k}, t)$). Since no ambiguity is likely to arise, and as commonly done, we omit any Fourier transform symbol to lighten notations.

Let us start with the full bubbly liquid's Lagrangian (3.6) and consider how its bubble cloud part L_b and its {bubble cloud - radiative field} part L_rb read within the framework of the continuous (*i.e.* macroscopic) description in the monodisperse-cloud case. We use expressions (2.3) of $m_{jj'}$, separate the $j = j'$ and $j \neq j'$ terms in the double sum in (3.6c) and then implement in (3.6c) and (3.6d) the substitutions

$$x_j(t) \rightsquigarrow X(\vec{r}, t) \quad \text{and} \quad x_{j'}(t) \rightsquigarrow X(\vec{r}\,', t), \tag{4.1a}$$

$$\sum_j R_j^3 \rightsquigarrow \frac{3f}{4\pi} \int \mathrm{d}^3 r \quad \text{and} \quad \sum_{j'} R_{j'}^3 \rightsquigarrow \frac{3f}{4\pi} \int \mathrm{d}^3 r'. \tag{4.1b}$$

All calculations carried out, we get

$$L_{\mathrm{b}} = \frac{3}{2} f \rho_w R^2 \Big\{ \int \mathrm{d}^3 r \Big[\Big(\frac{\partial X(\vec{r},t)}{\partial t} \Big)^2 - \omega_0^2 X^2(\vec{r},t) \Big] + \frac{k_{\mathrm{c}}^2}{4\pi} \iint \frac{\mathrm{d}^3 r\, \mathrm{d}^3 r'}{|\vec{r}-\vec{r}\,'|} \frac{\partial X(\vec{r},t)}{\partial t} \frac{\partial X(\vec{r}\,',t)}{\partial t} \Big\} \tag{4.2a}$$

and

$$L_{\mathrm{rb}} = \frac{3f}{4\pi} \rho_w \int \mathrm{d}^3 r \frac{\partial X(\vec{r},t)}{\partial t} \int \mathrm{d}^3 r' \frac{\partial \vec{U}_{\mathrm{r}}(\vec{r}\,',t)}{\partial t} . \frac{\vec{r}\,' - \vec{r}}{|\vec{r}\,' - \vec{r}|^3}. \tag{4.2b}$$

Defining the spatial Fourier transform $\varphi(\vec{k})$ of any field (scalar or vectorial) $\varphi(\vec{r})$ as

$$\varphi(\vec{k}) = \int \mathrm{d}^3 r\, \mathrm{e}^{-\mathrm{i}\vec{k}.\vec{r}} \varphi(\vec{r}), \tag{4.3}$$

and using the Parseval-Plancherel theorem, the three parts L_{r}, L_{b}, L_{rb} of the bubbly liquid's Lagrangian L respectively read

$$L_{\mathrm{r}} = \frac{1}{(2\pi)^3} \int \mathrm{d}^3 k\, \frac{1}{2} \rho_w \Big[\Big| \frac{\partial \vec{U}_{\mathrm{r}}(\vec{k},t)}{\partial t} \Big|^2 - c_w^2 \big| \vec{k}.\vec{U}_{\mathrm{r}}(\vec{k},t) \big|^2 \Big], \tag{4.4a}$$

$$L_{\mathrm{b}} = \frac{1}{(2\pi)^3} \int \mathrm{d}^3 k\, \frac{3}{2} f \rho_w R^2 \Big[\Big(1 + \frac{k_{\mathrm{c}}^2}{k^2} \Big) \Big| \frac{\partial X(\vec{k},t)}{\partial t} \Big|^2 - \omega_0^2 |X(\vec{k},t)|^2 \Big], \tag{4.4b}$$

$$L_{\mathrm{rb}} = \frac{1}{(2\pi)^3} \int \mathrm{d}^3 k\, 3 f \rho_w \frac{\partial X^*(\vec{k},t)}{\partial t} \Big(\frac{\mathrm{i}\vec{k}}{k^2} . \frac{\partial \vec{U}_{\mathrm{r}}(\vec{k},t)}{\partial t} \Big). \tag{4.4c}$$

The motion equations are easily derived from the above expressions. Substituting (by simple convenience) the above integrals $\frac{1}{(2\pi)^3} \int \mathrm{d}^3 k$ by the discrete summations $\frac{1}{\mathcal{V}} \sum_{\vec{k}}$ ($\mathcal{V}$ is here the total volume of the cloud), we get the conjugate momenta, index $\vec{k}$ being henceforth understood in all dynamical variables,

$$P_x = \frac{\partial L}{\partial \dot{X}} = \frac{\rho_w}{\mathcal{V}} \Big[3 f R^2 \Big(1 + \frac{k_{\mathrm{c}}^2}{k^2} \Big) \dot{X}^* - 3f \frac{\mathrm{i}\vec{k}.\dot{\vec{U}}_{\mathrm{r}}^*}{k^2} \Big], \tag{4.5a}$$

$$\vec{P}_{\vec{u}_{\mathrm{r}}} = \frac{\partial L}{\partial \dot{\vec{U}}_{\mathrm{r}}} = \frac{\rho_w}{\mathcal{V}} \Big[\dot{\vec{U}}_{\mathrm{r}}^* + 3f \frac{\mathrm{i}\vec{k}}{k^2} \dot{X}^* \Big]. \tag{4.5b}$$

The Lagrange's equations then read

$$\dot{P}_x = \frac{\partial L}{\partial X} \rightsquigarrow \ddot{X} + \omega_{\mathrm{b}}^2(\vec{k}) X = -\frac{1}{k^2 + k_{\mathrm{c}}^2} \frac{\mathrm{i}\vec{k}.\ddot{\vec{U}}_{\mathrm{r}}}{R^2}, \tag{4.6a}$$

$$\dot{\vec{P}}_{\vec{u}_{\mathrm{r}}} = \frac{\partial L}{\partial \vec{U}_{\mathrm{r}}} \rightsquigarrow \ddot{\vec{U}}_{\mathrm{r}} + c_w^2 (\vec{k}.\vec{U}_{\mathrm{r}}) \vec{k} = 3f \frac{\mathrm{i}\vec{k}}{k^2} \ddot{X}. \tag{4.6b}$$

The above set of coupled dynamical equations deserves a few comments. First, it is equivalent to the Fourier transform (for $|\vec{k}| \ll \frac{\pi}{d}$) of set (3.7). Second, it can be substituted by an equivalent set coupling X and P_{r}: owing to (3.5) which now reads

$$\rho_w \ddot{\vec{U}}_{\mathrm{r}} = -\mathrm{i}\vec{k} P_{\mathrm{r}}, \tag{4.7}$$

we have indeed

$$\ddot{X} + \omega_{\mathrm{b}}^2(\vec{k}) X = -\frac{k^2}{k^2 + k_{\mathrm{c}}^2} \frac{P_{\mathrm{r}}}{\rho_w R^2}, \tag{4.8a}$$

$$\ddot{P}_{\mathrm{r}} + \omega_{\mathrm{s}}^2(\vec{k}) P_{\mathrm{r}} = -3 f \rho_w \frac{\ddddot{X}}{k^2}, \tag{4.8b}$$

which is the Fourier transform (for $|\vec{k}| \ll \frac{\pi}{d}$) of set (3.8). Now, looking for a monochromatic solution $\{X(t) = \overline{X}\, \mathrm{e}^{-\mathrm{i}\omega t}$, $P_{\mathrm{r}}(t) = \overline{P}_{\mathrm{r}}\, \mathrm{e}^{-\mathrm{i}\omega t}\}$ of the above system, we immediately recover our bubble-glass dispersion relation (3.18), as expected.

Third, (4.6b) shows that the monochromatic radiative field $\vec{U}_\mathrm{r}$ is longitudinal, so that set (4.6) is easily turned into the equivalent form

$$\ddot{X} + \omega_0^2 X = \frac{c_w^2}{R^2}\mathrm{i}\vec{k}.\vec{U}_\mathrm{r}, \tag{4.9a}$$

$$\mathrm{i}\vec{k}.\ddot{\vec{U}}_\mathrm{r} + \omega_\mathrm{p}^2(\vec{k})\,\mathrm{i}\vec{k}.\vec{U}_\mathrm{r} = 3f\omega_0^2 X, \tag{4.9b}$$

where, index p in ω_p standing for "phonon",

$$\omega_\mathrm{p}^2(\vec{k}) = c_w^2(k^2 + k_\mathrm{c}^2). \tag{4.10}$$

Equation (4.9a) is remarkable from a threefold viewpoint.

(*i*) In any free motion, it links X and $\mathrm{i}\vec{k}.\vec{U}_\mathrm{r}$ *independently* of $\vec{k}$: in the $\vec{r}$-space, we have a *local* relation between X and $\mathrm{div}\,\vec{U}_\mathrm{r}$, namely (in time-Fourier transform)

$$X = \frac{c_w^2}{R^2}\frac{\mathrm{div}\,\vec{U}_\mathrm{r}}{\omega_0^2 - \omega^2}. \tag{4.11a}$$

(*ii*) Keeping nevertheless in mind that X and $\vec{U}_\mathrm{r}$ are macroscopic variables, equation (4.9a) and its solution (4.11a) describe the motion of a *single* bubble with radius R undergoing the *total* outer water extrapressure $P_w = -\frac{1}{\chi_w}\,\mathrm{div}\,\vec{U}_\mathrm{r}$. As already mentioned in section III about equation (3.5), P_w includes bubble-bubble instantaneous interactions as well as acoustic radiation, hence the absence of any $-\mathrm{i}\omega\Gamma$ radiative damping term in the denominator of the right-hand side of (4.11a).

(*iii*) The macroscopic relative air expansion $3X$ can thus be related to P_w, which leads to the ω-dependent compressibility

$$\chi_a(\omega) = \frac{3}{\rho_w R^2(\omega_0^2 - \omega^2)}. \tag{4.11b}$$

Owing for (1.4), we get $\chi_a(\omega) \simeq \chi_a$ for $\omega \ll \omega_0$, as expected. It is noteworthy that the cloud's compressibility is negative for $\omega > \omega_0$. This is not surprising: the Minneart bubble behaves like any harmonic oscillator, *i.e.* its response is 180° out of phase to any sollicitation at an angular frequency ω higher than its own (eigen) angular frequency ω_0. This means that, for $\omega > \omega_0$, the instantaneous (i) and retarded (r) parts of the pressure/displacement fields in splitting (3.4) are 180° out of phase with each other.

Moreover, it is noteworthy that, looking again for a monochromatic solution of (4.9), we are left with a new form of the dispersion relation

$$\left(\omega^2 - \omega_0^2\right)\left(\omega^2 - \omega_\mathrm{p}^2(\vec{k})\right) = c_w^2 k_\mathrm{c}^2 \omega_0^2, \tag{4.12}$$

which is equivalent to (3.18), as expected.

We would end the present subsection with the following remark. Although exact, the equation sets (4.6) or (4.8) do not provide the best possible description of the propagation modes as explained hereafter. Let us suppose, as we did in section II, that water in infinitely hard (*i.e.* $\chi_w = 0$) and consider equations (4.6a) or (4.8a): their right-hand sides vanish, because the radiative displacement $\vec{U}_\mathrm{r}$ (or pressure P_r) is then exactly zero; there is no sound wave; the dispersion curve is then reduced to the bubble wave branch. Now true water is almost incompressible compared to air: the bubble wave dispersion relation $\omega = \omega_\mathrm{b}(\vec{k})$ is consequently a good approximation of the lower branch $\omega = \omega_-(\vec{k})$ of the exact dispersion curve, as displayed by figure 2. The difference between $\omega_\mathrm{b}(\vec{k})$ and $\omega_-(\vec{k})$ lies only in their slopes at $k = 0$ (respectively $\frac{\omega_0}{k_\mathrm{c}}$ and $\frac{\omega_0}{k_\mathrm{c}}/\sqrt{1 + \frac{\omega_0^2}{c_w^2 k_\mathrm{c}^2}}$). Not so good is the accordance between the sound dispersion relation $\omega = \omega_\mathrm{s}(\vec{k}) = c_w|\vec{k}|$ and the upper branch $\omega = \omega_+(\vec{k})$ of the exact dispersion curve. To palliate this discrepancy, it is tempting to proceed as above and make the following symmetrical assumption: let us suppose that air is infinitely soft (*i.e.* $\chi_a = \infty$). The Minnaert angular frequency ω_0, and consequently $\omega_\mathrm{b}(\vec{k})$, vanish; there is no bubble wave; equation (4.6a) reduces to

$$X = X_\mathrm{s} = -\frac{1}{k^2 + k_\mathrm{c}^2}\frac{\mathrm{i}\vec{k}.\vec{U}_\mathrm{r}}{R^2}, \tag{4.13}$$

index s in X_s standing for "(infinitely) soft". Introducing (4.13) in (4.6b) gives *in fine*

$$\frac{\ddot{\vec{U}}_\mathrm{r}}{1+\frac{k_\mathrm{c}^2}{k^2}} + \omega_\mathrm{s}^2(\vec{k})\,\vec{U}_\mathrm{r} = 0 \quad \rightsquigarrow \quad \ddot{\vec{U}}_\mathrm{r} + \omega_\mathrm{p}^2(\vec{k})\,\vec{U}_\mathrm{r} = 0 \tag{4.14}$$

(which can be obtained as well by cancelling the right-hand side of (4.9b)). The sound wave propagation is then ruled by a dispersion relation $\omega_\mathrm{p}(\vec{k})$ of the Klein-Gordon type, displayed in dash-dotted line in figure 2. This is due to the fact that, even with infinitely compressible air inside, the bubbles pulsate and radiate an acoustic field which superimposes to the applied one, thus modifying propagation. This phenomenon can be regarded as an exotic example of acoustic diffraction, the importance of which is paradoxically maximum at large wavelengths ($k < k_\mathrm{c}$). Now true air is extremely compressible compared to water: the Klein-Gordon dispersion relation $\omega = \omega_\mathrm{p}(\vec{k})$ is consequently a good approximation of the upper branch $\omega = \omega_+(\vec{k})$ of the exact dispersion curve, as displayed by figure 2. The difference between $\omega_\mathrm{p}(\vec{k})$ and $\omega_+(\vec{k})$ lies only in the values of the cuttoffs at $k = 0$ (respectively $c_w k_\mathrm{c}$ and $\omega_0' = \sqrt{\omega_0^2 + c_w^2 k_\mathrm{c}^2}$).

In order to turn the above remark to profit, and as suggested by equations (4.13) and (4.14), let us make the following variables change:

$$X = X_\mathrm{r} + X_\mathrm{s}, \tag{4.15a}$$

$$\vec{U}_\mathrm{r} = \Big(1+\frac{k_\mathrm{c}^2}{k^2}\Big)\vec{U}_\mathrm{r}', \tag{4.15b}$$

index r in X_r standing for "resilient". Observe by the way the perfect symmetry between splittings (3.4b) and (4.15a): in the former splitting, $\vec{u}_\mathrm{i}$ is the infinitely hard water approximate of $\vec{u}$, and $\vec{u}_\mathrm{r}$ the corrective term accounting for true water's nonzero compressibility; in the latter splitting, X_s is the infinitely soft air approximation of X, and X_r the corrective term accounting for true air's noninfinite compressibility. Observe too that, despite the (misleading) simplicity of equation (4.15b), the transformation of $\vec{U}_\mathrm{r}$ into $\vec{U}_\mathrm{r}'$ is far from trivial: it is local in the $\vec{k}$-space (because of our assuming the bubble cloud's translational macroscopic invariance), but nonlocal in the usual $\vec{r}$-space in which we have

$$\vec{U}_\mathrm{r}'(\vec{r},t) = \int \mathrm{d}^3 s\, \vec{U}_\mathrm{r}(\vec{r}-\vec{s},t)\, g(\vec{s}), \tag{4.16a}$$

with, δ standing for the Dirac function,

$$g(\vec{s}) = \delta(\vec{s}) - \frac{k_\mathrm{c}^2}{4\pi|\vec{s}\,|}\mathrm{e}^{-k_\mathrm{c}|\vec{s}\,|}. \tag{4.16b}$$

Observe at last that the new set $\{X_\mathrm{r}, \vec{U}_\mathrm{r}'\}$ of dynamical variables provides a new version of splitting (3.4b), which reads in Fourier transform:

$$\vec{U} = \vec{U}_\mathrm{r} - 3f\frac{\mathrm{i}\vec{k}}{k^2}X = \vec{U}_\mathrm{r}' - 3f\frac{\mathrm{i}\vec{k}}{k^2}X_\mathrm{r} \quad (|\vec{k}\,| \ll \frac{\pi}{d}), \tag{4.17}$$

as can be checked using (4.13). In this new set of variables, and owing to expressions (4.4), the bubbly liquid's Lagrangian reads

$$L = \frac{1}{\mathcal{V}}\sum_{\vec{k}} \rho_w\Big(1+\frac{k_\mathrm{c}^2}{k^2}\Big)\left\{\frac{3}{2}fR^2\left[|\dot{X}_\mathrm{r}|^2 - \omega_\mathrm{b}^2(\vec{k})|X_\mathrm{r}|^2\right] + \frac{1}{2}\left[|\dot{\vec{U}}_\mathrm{r}'|^2 - \omega_\mathrm{u}^2(\vec{k})|\vec{U}_\mathrm{r}'|^2\right] + 3f\omega_\mathrm{b}^2(\vec{k})X_\mathrm{r}^*\frac{\mathrm{i}\vec{k}.\vec{U}_\mathrm{r}'}{k^2}\right\}, \tag{4.18a}$$

where, index u in ω_u standing for "upper (branch)",

$$\omega_\mathrm{u}^2(\vec{k}) = \omega_\mathrm{p}^2(\vec{k}) + \frac{k_\mathrm{c}^2}{k^2}\omega_\mathrm{b}^2(\vec{k}) = c_w^2(k^2+k_\mathrm{c}^2) + \omega_0^2\frac{k_\mathrm{c}^2}{k^2+k_\mathrm{c}^2}. \tag{4.18b}$$

The $\omega_\mathrm{u}(\vec{k})$ curve is displayed in dotted line in figure 2. As a consequence of (4.18a), the conjugate momenta of X_r and $\vec{U}_\mathrm{r}'$ are

$$P_{x_\mathrm{r}} = \frac{\partial L}{\partial \dot{X}_\mathrm{r}} = \frac{\rho_w}{\mathcal{V}}3fR^2\Big(1+\frac{k_\mathrm{c}^2}{k^2}\Big)\dot{X}_\mathrm{r}^*, \tag{4.19a}$$

$$\vec{P}_{\vec{u}_\mathrm{r}'} = \frac{\partial L}{\partial \dot{\vec{U}}_\mathrm{r}'} = \frac{\rho_w}{\mathcal{V}}\Big(1+\frac{k_\mathrm{c}^2}{k^2}\Big)\dot{\vec{U}}_\mathrm{r}'^{*}, \tag{4.19b}$$

to be compared with expressions (4.5). It is noteworthy that the {bubble cloud - radiative field} bilinear coupling in (4.18a) is *elastic* (product $X_{\rm r}^* \vec{U}'_{\rm r}$) whereas it was *inertial* in (4.4c) (product $\dot{X}^* \dot{\vec{U}}_r$). With the new variables, the motion equations (4.6) now read

$$\dot{P}_{x_{\rm r}} = \frac{\partial L}{\partial X_{\rm r}} \quad \rightsquigarrow \quad \ddot{X}_{\rm r} + \omega_{\rm b}^2(\vec{k})\, X_{\rm r} = \frac{\omega_{\rm b}^2(\vec{k})}{k^2 R^2} {\rm i}\vec{k}.\vec{U}'_{\rm r}, \tag{4.20a}$$

$$\dot{\vec{P}}_{\vec{u}'_{\rm r}} = \frac{\partial L}{\partial \vec{U}'_{\rm r}} \quad \rightsquigarrow \quad \ddot{\vec{U}}'_{\rm r} + \omega_{\rm u}^2(\vec{k})\, \vec{U}'_{\rm r} = -3f \frac{{\rm i}\vec{k}}{k^2} \omega_{\rm b}^2(\vec{k}) X_{\rm r}. \tag{4.20b}$$

Looking for a monochromatic solution of the above system, we get a new version of the dispersion relation, namely

$$\left(\omega^2 - \omega_{\rm b}^2(\vec{k})\right)\left(\omega^2 - \omega_{\rm u}^2(\vec{k})\right) = \omega_{\rm b}^4(\vec{k}) \frac{k_{\rm c}^2}{k^2}, \tag{4.21}$$

which is of course equivalent to (3.18) and (4.12). The main advantage of the new couple $\{X_{\rm r}, \vec{U}'_{\rm r}\}$ of dynamical variables over the ancient one $\{X, \vec{U}_{\rm r}\}$ is that $\omega = \omega_{\rm u}(\vec{k})$ is a much better approximation of the exact dispersion curve's upper branch $\omega = \omega_+(\vec{k})$ than $\omega = \omega_{\rm s}(\vec{k})$, as obvious from figure 2. It is noteworthy indeed that angular frequencies $\omega_{\rm u}(\vec{k})$ and $\omega_+(\vec{k})$ have the same value $\omega'_0 = \sqrt{\omega_0^2 + c_w^2 k_{\rm c}^2}$ for $k = 0$, and the same asymptotic behaviour for $k \gg k_{\rm c}$, hence their great resemblance. Let us note too that $\omega_{\rm u}$ coincides with $\omega_{\rm p}$ for $\chi_a \to \infty$.

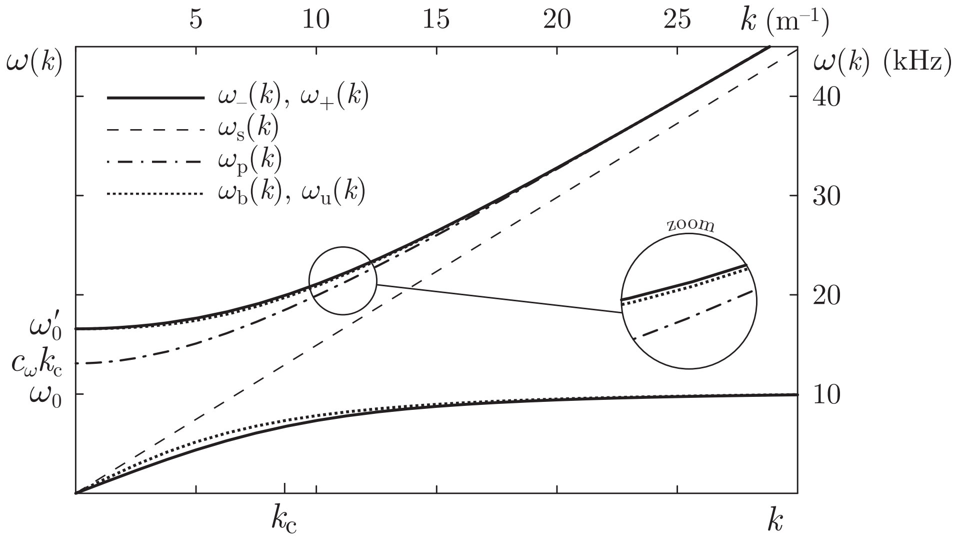

FIG. 2: The different dispersion relations $\omega(k)$ introduced in the text. In solid lines, both branches $\omega_\pm(k)$ of the exact dispersion relation (as in figure 1). In dashed line, the sound dispersion relation $\omega_{\rm s}(k) = c_w k$. In dash-dotted line, the Klein-Gordon dispersion relation $\omega_{\rm p}(k) = c_w \sqrt{k^2 + k_{\rm c}^2}$, which corresponds to the limit-case $\chi_a = \infty$. In dotted lines, the best approximations we could find of the exact dispersion relation: (*i*) $\omega_{\rm u}(k)$ for the upper branch $\omega_+(k)$; (*ii*) $\omega_{\rm b}(k)$, which corresponds to the limit-case $\chi_w = 0$, for the lower branch $\omega_-(k)$. The numerical values are the same as in figure 1.

In the next section, we shall discuss two additional aspects we have disregarded hitherto: how are the propagation modes excited? how are they damped?

V. PROPAGATIVE MODES: EXCITATION AND DAMPING

A. Excitation

In pure water, sound waves are excited by means of some external pressure source (loud-speaker or other device). More precisely, an applied extrapressure field $p_{\rm ext}(\vec{r}, t)$ is equivalent, as concerns the fluid's motion, to a longitudinal external force density

$$\vec{\mathcal{F}}(\vec{r}, t) = -\vec{\nabla} p_{\rm ext}(\vec{r}, t) \tag{5.1}$$

which acts as a source term in the d'Alembert sound equation:

$$\rho_w \frac{\partial^2 \vec{u}}{\partial t^2} - \frac{1}{\chi_w} \overrightarrow{\Delta}\, \vec{u} = \vec{\mathcal{F}}. \tag{5.2}$$

Now in bubbly water, we have shown that propagation should be described by the twofold set $\{x_j, \vec{u}_{\rm r}\,(\text{resp. } p_{\rm r})\}$ of dynamical variables, which can be averaged at the continuous (diphasic) medium approximation (*i.e.* in the center of the Brillouin zone) in a macroscopic twofold field $\{X, \vec{U}_{\rm r}\,(\text{resp. } P_{\rm r})\}$. In the present subsection, we address the following question: how can we determine this twofold field when we know $\vec{\mathcal{F}}$?

As a matter a fact, the answer is quite simple. To begin with, let us consider Lagrangian (4.18a) again, to which we add the external term $L_{\rm ext}(t) = \int \mathrm{d}^3 r\, \vec{\mathcal{F}}(\vec{r}, t).\vec{U}$. Owing for (4.17) and for the Parseval-Plancherel theorem, this additional term reads (Fourier transform symbol and index $\vec{k}$ in dynamical variables being understood)

$$L_{\rm ext} = \frac{1}{\mathcal{V}} \sum_{\vec{k}} \vec{\mathcal{F}}^*(\vec{k}, t).\Big[\vec{U}'_{\rm r} - 3f \frac{\mathrm{i}\vec{k}}{k^2} X_{\rm r}\Big]. \tag{5.3}$$

Then the derived Lagrange's equation can be written in the matricial form

$$\begin{bmatrix} \frac{\partial^2}{\partial t^2} + \omega_{\rm b}^2(\vec{k}) & -\frac{\omega_{\rm b}^2(\vec{k})}{k^2 R^2} \\ -3f\omega_{\rm b}^2(\vec{k}) & \frac{\partial^2}{\partial t^2} + \omega_{\rm u}^2(\vec{k}) \end{bmatrix} \begin{bmatrix} X_{\rm r} \\ \mathrm{i}\vec{k}.\vec{U}'_{\rm r} \end{bmatrix} = \frac{\mathrm{i}\vec{k}.\vec{\mathcal{F}}(\vec{k}, t)}{\rho_w \big(1 + \frac{k_{\rm c}^2}{k^2}\big)} \begin{bmatrix} \frac{1}{k^2 R^2} \\ 1 \end{bmatrix}, \tag{5.4}$$

which generalizes (4.20). The above system is easily solved in the $(\vec{k}, \omega)$ Fourier representation. We find

$$\begin{bmatrix} X_{\rm r} \\ \mathrm{i}\vec{k}.\vec{U}'_{\rm r} \end{bmatrix} = \frac{1}{D(\vec{k}, \omega)} \begin{bmatrix} -\omega^2 + \omega_{\rm u}^2(\vec{k}) & \frac{\omega_{\rm b}^2(\vec{k})}{k^2 R^2} \\ 3f\omega_{\rm b}^2(\vec{k}) & -\omega^2 + \omega_{\rm b}^2(\vec{k}) \end{bmatrix} \begin{bmatrix} \frac{1}{k^2 R^2} \\ 1 \end{bmatrix} \times \frac{\mathrm{i}\vec{k}.\vec{\mathcal{F}}(\vec{k}, \omega)}{\rho_w \big(1 + \frac{k_{\rm c}^2}{k^2}\big)}, \tag{5.5a}$$

with

$$D(\vec{k}, \omega) = \big(\omega^2 - \omega_{\rm b}^2(\vec{k})\big)\big(\omega^2 - \omega_{\rm u}^2(\vec{k})\big) - \frac{k_{\rm c}^2}{k^2}\omega_{\rm b}^4(\vec{k}). \tag{5.5b}$$

It is noteworthy that the above determinant can be factorized in

$$D(\vec{k}, \omega) = c_w^2 \big(\omega_0^2 - \omega^2\big)\big(k^2 - k^2(\omega)\big), \tag{5.5c}$$

where

$$k^2(\omega) = \frac{\omega^2}{c_w^2} \frac{\omega_0'^2 - \omega^2}{\omega_0^2 - \omega^2} = k_{\rm s}^2 + 4\pi n \mathcal{S} \tag{5.5d}$$

is the (squarred) damping-free Foldy wave vector in a monodisperse bubble cloud (see (1.2), (1.3)).

Next, using definitions (4.11b), (4.13), (4.15a), (4.15b), factorization (5.5c) and result (4.17), we end up with

$$X = X_{\rm r} + X_{\rm s} = \frac{1}{3}\chi_a(\omega) \frac{\mathrm{i}\vec{k}.\vec{\mathcal{F}}(\vec{k}, \omega)}{k^2 - k^2(\omega)}, \tag{5.6a}$$

$$\vec{U}_{\rm r} = \Big(1 + \frac{k_{\rm c}^2}{k^2}\Big)\vec{U}'_{\rm r} = \chi_w \frac{\vec{\mathcal{F}}(\vec{k}, \omega)}{k^2 - k^2(\omega)}, \tag{5.6b}$$

$$\vec{U} = \vec{U}_{\rm r} - 3f\frac{\mathrm{i}\vec{k}}{k^2} X = \chi_w \frac{\omega_0'^2 - \omega^2}{\omega_0^2 - \omega^2} \frac{\vec{\mathcal{F}}(\vec{k}, \omega)}{k^2 - k^2(\omega)}. \tag{5.6c}$$

Observe that, considering (5.6a) and (5.6b), relation (4.11a) is generalized to a forced motion. Observe too that expression (5.6c) reads exactly as it would in an effective fluid with mass density ρ_w and compressibility

$$\chi_{\rm eff}(\omega) = \chi_w \frac{\omega_0'^2 - \omega^2}{\omega_0^2 - \omega^2} = \chi_w + f\chi_a(\omega). \tag{5.7a}$$

The above result is of the utmost interest: as far as the *full* displacement field $\vec{U}$ is concerned, the response of a bubbly water (with $f \ll 1$) to an external macroscopic pressure force is that of an *effective* fluid with mass density

ρ_w and compressibility $\chi_{\text{eff}}(\omega)$. This is just the generalization of the Wood's analysis we outlined in introduction. Compressibility $\chi_{\text{eff}}(\omega)$ is negative for $\omega_0 < \omega < \omega'_0$, hence the forbidden band. As expected, this effective fluid coincides with pure water in the $f = \frac{4}{3}\pi n R^3 \to 0$ limit. In this limit indeed, $k_{\text{c}} = 4\pi n R \to 0$ implying $\omega'_0 \to \omega_0$ and $\chi_{\text{eff}}(\omega) \to \chi_w$ (see (5.7a)) and, allowing for (4.15b) through (4.17), $\vec{U}'_{\text{r}}$ and $\vec{U}_{\text{r}}$ do coincide *in fine* with the full water displacement field $\vec{U}$.

Observe at last that, comparing (5.6b) and (5.6c), we get

$$\vec{U} = \frac{\omega_0'^2 - \omega^2}{\omega_0^2 - \omega^2} \vec{U}_{\text{r}}, \tag{5.7b}$$

which means that the relation between $\vec{U}_{\text{r}}$ and $\vec{U}$ is *local* in the $\vec{r}$-space. This allows to define an ω-dependent acoustic refractive index of the effective fluid by

$$n^2(\omega) = \frac{\chi_{\text{eff}}(\omega)}{\chi_w} = \frac{\omega_0'^2 - \omega^2}{\omega_0^2 - \omega^2}. \tag{5.7c}$$

We would now end the present subsection with the exact calculation of the twofold monochromatic field $\{X, \vec{U}_{\text{r}}\}$ propagation's Green function. With this aim, we consider a pulsating sphere with radius R_{s} larger than d but smaller than the emitted wavelength, and located at the origin of coordinates. Let $p_{\text{e}}(t)$ be the external pressure imposed inside this sphere. Applying (5.1) with

$$p_{\text{ext}}(\vec{r}, t) = p_{\text{e}}(t)\, \theta(R_{\text{s}} - r) \tag{5.8a}$$

(θ is the Heaviside function), we have

$$\vec{\mathcal{F}}(\vec{r}, t) = p_{\text{e}}(t)\, \delta(r - R_{\text{s}})\, \vec{e}_r \quad \rightsquigarrow \quad \vec{\mathcal{F}}(\vec{k}, t) = -\mathrm{i}\vec{k}\, E_{\text{e}}(t) \quad (kR_{\text{s}} \ll 1), \tag{5.8b}$$

with

$$E_{\text{e}}(t) = \frac{4}{3}\pi R_{\text{s}}^3 p_{\text{e}}(t). \tag{5.8c}$$

Then, using equations (5.6), we get

$$X(\vec{k}, \omega) = \frac{1}{3}\chi_a(\omega) \frac{k^2 E_{\text{e}}(\omega)}{k^2 - k^2(\omega)}, \tag{5.9a}$$

$$\vec{U}_{\text{r}}(\vec{k}, \omega) = \chi_w \frac{-\mathrm{i}\vec{k} E_{\text{e}}(\omega)}{k^2 - k^2(\omega)}, \tag{5.9b}$$

$$\vec{U}(\vec{k}, \omega) = \chi_{\text{eff}}(\omega) \frac{-\mathrm{i}\vec{k} E_{\text{e}}(\omega)}{k^2 - k^2(\omega)}. \tag{5.9c}$$

The above results read, in the $\vec{r}$-space,

$$X(\vec{r}, \omega) = -\frac{1}{3}\chi_a\, \Delta\, \Phi(\vec{r}, \omega), \tag{5.10a}$$

$$\vec{U}_{\text{r}}(\vec{r}, \omega) = -\chi_w \vec{\nabla}\Phi(\vec{r}, \omega), \tag{5.10b}$$

$$\vec{U}(\vec{r}, \omega) = -\chi_{\text{eff}}(\omega) \vec{\nabla}\Phi(\vec{r}, \omega), \tag{5.10c}$$

with

$$\Phi(\vec{r}, \omega) = \frac{E_{\text{e}}(\omega)}{(2\pi)^3} \int \mathrm{d}^3k \frac{\mathrm{e}^{\mathrm{i}\vec{k}.\vec{r}}}{k^2 - k^2(\omega)} = \frac{E_{\text{e}}(\omega)}{4\pi r} \mathrm{e}^{\mathrm{i}k(\omega) r}. \tag{5.10d}$$

The above results (5.10) mean that the three fields X, $\vec{U}_{\text{r}}$ and $\vec{U}$ can be derived from a unique scalar source potential Φ. Of course, in the case of an extended external pressure source $p_{\text{ext}}(\vec{r}, t)$, the above expressions (5.10a through c) still hold, with expression (5.10d) of the source potential simply generalized in

$$\Phi(\vec{r}, \omega) = \int \mathrm{d}^3 s\, \frac{p_{\text{ext}}(\vec{r}, \omega)}{4\pi |\vec{r} - \vec{s}\,|} \mathrm{e}^{\mathrm{i}k(\omega)|\vec{r} - \vec{s}\,|}. \tag{5.10e}$$

B. Damping

So far, we have neglected dissipation. In fact, even in pure water, sound waves are attenuated. Nevertheless it can be shown that, in bubbly liquids, most of the dissipation is concentrated inside the bubbles or at the air-water interface. Hereafter, we shall briefly recall Prosperetti's approach [6] and use it to complete our description of the propagation. Roughly, Prosperetti distinguishes two kinds of dissipation processes: viscous and thermal.

The former process originates in water's viscosity (in comparison, air's viscosity is negligible), so that, strictly speaking, expression (3.1) (for instance) should be revised, using the Navier-Stokes equation instead of Newton's law. Practically, the viscous stress results in an additional extrapressure $p^{\mathrm{vis}} = 4\eta_w \dot{x}$ (η_w is the water first viscosity coefficient) exerted upon each bubble's surface, which *in fine* finds expression in a viscous damping rate $\Gamma^{\mathrm{vis}} = \frac{4\eta_w}{\rho_w R_j^2}$ in equation (2.7a) (and in all equations ruling bubble j's dynamics).

The latter process originates in thermal exchanges at the bubbles' surface between water (regarded as a heat reservoir owing to its high heat capacity) and air (that undergoes temperature oscillations). This results in an ω-dependent Minnaert angular frequency $\omega_0(\omega)$ and an ω-dependent thermal damping rate $\Gamma^{\mathrm{th}}(\omega)$.

As a consequence of both above dissipation processes, equation (2.7a) reads, in time-Fourier transform:

$$-\,\omega^2 x_j - \sum_{j'\neq j} \frac{R}{|\vec{r}_j - \vec{r}_{j'}|}\omega^2 x_{j'} - \mathrm{i}\omega\Gamma^{\mathrm{eff}}(\omega)x_j + \omega_0^2(\omega)x_j = 0, \tag{5.11}$$

with

$$\Gamma^{\mathrm{eff}}(\omega) = \Gamma^{\mathrm{vis}} + \Gamma^{\mathrm{th}}(\omega). \tag{5.12}$$

Looking for a solution of (5.11) of the form (2.6), we are left with the implicit equation

$$\omega^2 = \omega_{\mathrm{b}}^2(\vec{k},\omega) = \frac{\omega_0^2(\omega) - \mathrm{i}\omega\Gamma^{\mathrm{eff}}(\omega)}{1 + \sum_{\vec{s}\neq 0} \frac{R}{|\vec{s}|}\mathrm{e}^{\mathrm{i}\vec{k}.\vec{s}}}, \tag{5.13}$$

the solution of which gives a *complex* angular frequency for the bubble wave with the *real* wave vector $\vec{k}$. Not surprisingly, dissipation results in a finite lifetime for the bubble waves. It is noteworthy that the above equation (5.13) is obtained from (2.7b) by simply changing ω_0^2 in $\omega_0^2(\omega) - \mathrm{i}\omega\Gamma^{\mathrm{eff}}(\omega)$, and that the motion equations ruling the radiative field's dynamics are not affected by dissipation. It is therefore rather easy to generalize the dispersion relation (3.18) for example, or its factorization (5.5c), so that (5.5d) ultimately reads

$$k^2(\omega) = k_{\mathrm{s}}^2(\omega) + \frac{4\pi n R\,\omega^2}{\omega_0^2(\omega) - \omega^2 - \mathrm{i}\omega\Gamma^{\mathrm{eff}}(\omega)}. \tag{5.14}$$

We are thus left with the fascinating following conclusion: Foldy's formula (1.3) is recovered, provided that the radiative damping rate Γ be substituted by the { viscous - thermal } effective damping rate $\Gamma^{\mathrm{eff}}(\omega)$, and that ω_0^2 be substituted by $\omega_0^2(\omega)$.

In the particular case of a monochromatic excitation at *real* angular frequency ω of the bubbly liquid's propagative modes, wave vector $k(\omega)$ given by (5.14) is *complex* and reads

$$k(\omega) = k'(\omega) + \mathrm{i}\,k''(\omega). \tag{5.15}$$

We have displayed the real $k'(\omega)$ and imaginary $k''(\omega)$ parts of $k(\omega)$ in figure 3 (a and b). For comparison purposes, we have first used Foldy's scattering function (1.3) with Γ gathering radiative, viscous and thermal dampings (dashed lines), then our own calculation (5.14) including (solid lines) or excluding (dotted lines) dissipation. Not surprisingly, Foldy predicts a higher k' in the forbidden band, and a higher k'' elsewhere. Confronting these theoretical predictions with experimental data is of course an issue of the utmost necessity. It needs a detailled and careful discussion, far beyond the scope of the present paper. We postpone it for a further study and summarize our results below.

VI. CONCLUSION

The acoustic bubble is a puzzling object, essentially because it is *not*, strictly speaking, *localized.* Whereas its stiffness is due to the gas compressibility (plus the gas-liquid surface tension for small bubbles), and can be considered

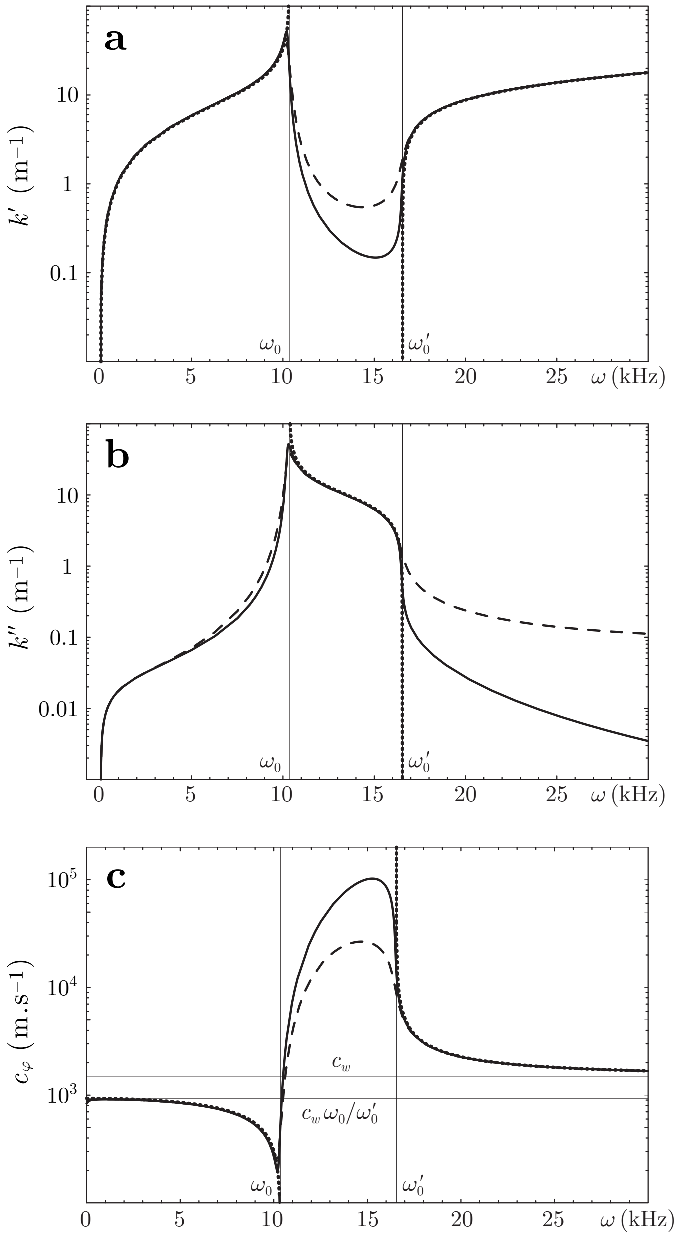

FIG. 3: Real $k'(\omega)$ (a) and imaginary $k''(\omega)$ (b) parts of the complex wave vector $k(\omega)$, as explained in the text. In (c), the phase velocity $c_\varphi = \omega/k'(\omega)$. In respectively dashed and solid lines, Foldy's prediction and our own calculation, both taking dissipation into account; in dotted lines, our calculation with $\Gamma^{\text{eff}}(\omega) = 0$, *i.e.* neglecting dissipation. The numerical values are those of figures 1 and 2. The dissipation is calculated following Prosperetti's analysis [6].

as localized *inside* (or at the surface of) the bubble, its mass is due to the liquid's inertia, and is therefore distributed *outside* the bubble. Hence the difficulty, when a bubble interacts with a pressure wave, to analyse the fluid's motion and tell what part is the bubble's breathing and what part is the sound. The situation is even more intricate in presence of a N-bubble cloud: the bubbles' inertia is then described by a $N \times N$ matrix, and any incoming pressure wave interacts with the whole cloud at a time. The Isolated Scattering Appproximation (ISA) is not available and *stricto sensu* the individual scattering of the wave by a given bubble – and consequently the concept itself of scattering path – make no sense.

To tackle this difficulty, we propose the radiative picture analysis. Within this framework, the bubbly liquid's motion is described by a twofold set of dynamical variables: the discrete set $\{x_j\}$ fot the bubble cloud, the continuous radiative field $\vec{u}_\text{r}$ (p_r) for the proper degrees of freedom of the liquid. Set $\{x_j\}$'s and field $\vec{u}_\text{r}$ (p_r)'s dynamics are coupled together. Thanks to this description, we have been able to derive an exact expression for the dispersion relation of a pressure wave propagating in a regular array of (like) bubbles, available throughout any entire Brillouin zone and including all Bragg-like process.

At wavelengths large compared to the bubble-bubble mean nearest-neighbour distance d, set $\{x_j\}$ and field $\vec{u}_{\rm r}$ $(/p_{\rm r})$ can be substituted by a macroscopic twofold field $\{X, \vec{U}_{\rm r}(/P_{\rm r})\}$. It is noteworthy that our description of bubbly liquids is not without a certain resemblance with Biot's theory of porous media [26]. The main difference with this theory lies in the coupling between both fields: in Biot, the liquid and the solid are coupled through viscosity, whereas our variables X and $\vec{U}_{\rm r}$ are inertially coupled.

With the aim of approaching the dispersion relation of the pressure waves through bubbly liquids, two limit cases are of interest. First, the incompressible liquid limit ($\chi_w = \infty$). In this limit, the radiative field is zero, *i.e.* the liquid's motion is but the instantaneous accompaniment of the bubbles' breathings. The large-wavelength modes of the bubble cloud, referred to as bubbles waves, or simply "bubblons", can propagate in the low-pass band $[0, \omega_0]$. The second limit case of interest is the infinitely soft gas limit ($\chi_a = 0$). In this limit, the bubbles' breathing is reduced to a simple accompaniment of the radiative motion (see (4.13)), without any resilience. The large-wavelength modes of this "hollow liquid", referred to as sound waves, or simply "phonons", can propagate only at angular frequencies higher than the cutoff $c_w k_{\rm c}$. Due to the (soft) bubble-holes in the liquid, the dispersion relation is no longer the nondispersive $\omega_{\rm s}(k)$ one, but the dispersive $\omega_{\rm p}(k)$ Klein-Gordon one. In the words of special relativity, one could say that the soft holes provide the phonon with a mass $m_{\rm p} = \hbar k_{\rm c}/c_w$.

In a real bubbly liquid, neither χ_w is zero, nor χ_a infinite. This results in the existence of a two-branch dispersion curve displayed in solid lines in figures 1 and 2. As can be checked from figure 2, the bubblon curve $\omega_{\rm b}(k)$ (lower dotted line) and the phonon curve $\omega_{\rm p}(k)$ (dash-dotted line) do repel each other in the $k \simeq 0$ region, thus becoming the respectively lower and upper branches of the exact dispersion curve. This level anticrossing, which lowers the slope of the bubblon curve and raises the cutoff of the phonon curve, is characteristic of a bubblon-phonon interaction. Amazingly, our study of the dispersion curve ends up with the puzzling conclusion that it coincides with Foldy's one provided that the radiative damping is withdrawn from the latter. This coincidence originates in the corner-stone macroscopic equation (4.9a), which means that Foldy's analysis can be applied, *as if* the ISA was available. In this average, any individual radiative damping is "washed out" by destructive interferences.

In this paper, we have also examined the problem of the twofold field $\{X, \vec{U}_{\rm r}(/P_{\rm r})\}$'s excitation by acoustic sources. We have shown that, as far as this excitation is implemented by means of a macroscopic longitudinal force (*i.e.* an outer extrapressure gradient varying slowly on a length scale of order of d), and the total macroscopic displacement field $\vec{U}$ is regarded, the bubbly liquid behaves as an effective fluid with mass density ρ_w and ω-dependent compressibility $\chi_{\rm eff}(\omega)$.

At last, we have considered dissipation and shown that it results in a finite lifetime of the bubblons. The dispersion relation can be modified accordingly, yielding in forced regime at given angular frequency ω, an ω-dependent complex wave vector $k(\omega)$ which coincides *mutatis mutandis* with Foldy's result.

We would emphasize that our study is far from exhaustive: we have considered here only the case of monodisperse clouds and large wavelengths. An extension to polydisperse clouds, smaller wavelengths, as well as a meticulous comparison with experimental results is of course needed.

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